A.3.1 Tensor invariance

Let's presume that x^i is the old Cartesian coordinate system, and $\tilde{x}^j$ represents the new curvilinear coordinate system. Both systems are related by transformation rules (A.4) and (A.10). The distance between the material points should be *invariant*, i.e. independent on the coordinate system, thus:

$$dl^2 = dx^i dx^i = b^i_j d\tilde{x}^j b^i_k d\tilde{x}^k = b^i_j b^i_k d\tilde{x}^j d\tilde{x}^k \tag{A.34}$$

Definition A.3.1 **Metric Tensor**

The metric tensor*, g_{ij} is defined from the relation (A.34) as:*

$$g_{jk} \equiv b^i_j b^i_k = \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial x^i}{\partial \tilde{x}^k} \tag{A.35}$$

The metric tensor is also called the fundamental tensor.

Using (A.35), we can rewrite (A.34) as:

$$dl^2 = g_{ij} d\tilde{x}^i d\tilde{x}^j \tag{A.36}$$

The inverse of the metric tensor is also called the conjugate metric tensor, g^{ij}, which satisfies the relation:

$$g^{ik} g_{kj} = \delta_{ij} \tag{A.37}$$

(see Problem 2.7.7).

Definition A.3.2 **Conjugate tensors**

For each index of a tensor we introduce the conjugate tensor where this index is transfered to its counterpart (covariant/contravariant) using the relations:

$$\begin{aligned} A^i &= g^{ij} A_j \\ A_i &= g_{ij} A^j \end{aligned} \tag{A.38}$$

Conjugate tensor is also called the *associate tensor*. Relations (A.38) are also called as operations of *raising/lowering of indexes* (Problem A.4.6).

Remark A.3.3 Tensor invariance

Since the transformation rules defined by (A.1.1) have a simple multiplicative character, any tensor expression should retain it's original form under transformation into a new coordinate system. Thus if an expression is given in a tensor form it will be invariant *under coordinate transformations.*

Not all the expressions constructed from tensor terms in curvilinear coordinates will be tensors themselves. For example, if vectors A_i and B_i are tensors, then A_iB_i is not generally a tensor[3]. However, if we consider the same operation on a contravariant tensor A^i and a covariant tenso B_i then the product will form an invariant:

$$\overline{A}^i\overline{B}_i = A^iB_i \tag{A.39}$$

Thus in curvilinear coordinates we have to refine the definition of the scalar product (Corollary A.2.12) or the index contraction operation to make it invariant (Problem A.4.5).

Definition A.3.4 Invariant Scalar Product

The invariant form *of the scalar product between two covariant vectors* A_i *and* B_i *is* $g^{ij}A_iB_j$*. Similarly, the invariant form of a scalar product between two contravariant vectors* A^i *and* B^i *is* $g_{ij}A^iB^j$*, where* g_{ij} *is the metric tensor (A.35) and* g^{ij} *is its conjugate (A.37).*

Corollary A.3.5 Two forms of a scalar product

According to (A.38) the scalar product can be represented by two invariant forms: A^iB_i *and* A_iB^i *(see Problem A.4.7).*

A simple scalar value, S, is invariant under coordinate transformations. A partial derivative of an invariant is a first order covariant tensor (vector):

$$A^i = S_{,i} = \frac{\partial S}{\partial x^i}$$

However, a partial derivative of a tensor of the order one and greater is not generally an invariant under coordinate transformations of type (A.6) and (A.7).

---

[3] For Cartesian tensors any product of tensors will always be a tensor, but this is not so for general tensors

A.3.2 Covariant differentiation

In curvilinear coordinate system we should use more complex differentiation rules to preserve the invariance of the derivative. These rules are called the rules of *covariant differentiation* and they guarantee that the derivative itself is a tensor. According to these rules the derivatives for covariant and contravariant indices will be slightly different. They are expressed as follows:

$$A_{i,j} \equiv \frac{\partial A_i}{\partial x^j} - \left\{ {k \atop ij} \right\} A_k \tag{A.40}$$

$$A^i_{,j} \equiv \frac{\partial A^i}{\partial x^j} + \left\{ {i \atop kj} \right\} A^k \tag{A.41}$$

where the contstruct $\left\{ {k \atop ij} \right\}$ is defined as

$$\left\{ {k \atop ij} \right\} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

and is also known in tensor calculus as *Christoffel's symbol* of the second kind [8]. Tensor g^{ij} represents the inverse of the metric tensor g_{ij} (A.37). As can be seen differentiation of a single component of a vector will involve all other components of this vector.

In differentiating higher order tensors each index should be treated independently. Thus differentiating a second order tensor, A^{ij}, should be performed as:

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \{ {m \atop ik} \} A_{mj} - \{ {m \atop jk} \} A_{im}$$

and as can be seen also involves all the components of this tensor. Likewise for the contravariant second order tensor A^{ij} we have:

$$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \{ {i \atop mk} \} A^{mj} + \{ {j \atop mk} \} A^{im} \tag{A.42}$$

And for a general n-covariant, m-contravariant tensor we have:

$$\begin{aligned} A^{j_1..j_m}_{i_1..i_n,p} &= \frac{\partial}{\partial x^p} A^{j_1..j_m}_{i_1..i_n,k} \\ &+ \{\genfrac{}{}{0pt}{}{j_1}{qp}\} A^{q j_2..j_m}_{i_1..i_n} + \cdots + \{\genfrac{}{}{0pt}{}{j_m}{qp}\} A^{j_1..j_{m-1} q}_{i_1..i_n} \\ &+ \{\genfrac{}{}{0pt}{}{q}{i_1 p}\} A^{j_1..j_m}_{q i_2..i_n} + \cdots + \{\genfrac{}{}{0pt}{}{q}{i_n p}\} A^{j_1..j_m}_{i_1..i_{n-1} q} \end{aligned} \tag{A.43}$$

Despite their seeming complexity, the relations of covariant differentiation can be easily implemented algorithmically and used in numerical solutions on arbitrary curved computational grids (Problem A.4.9).

Definition A.3.6 Divergence

Divergence of a vector is defined as $A^i_{,i}$*:*

$$divA \equiv A^i_{,i} \tag{A.44}$$

From this definition and the rule of covariant differentiation (A.41) we have:

$$A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \{\genfrac{}{}{0pt}{}{i}{ki}\} A^k \tag{A.45}$$

this can be shown [7] to be equal to:

$$\begin{aligned} A^i_{,i} &= \frac{\partial A^i}{\partial x^i} + \left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g} \right) A^i \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} A^i \right) \end{aligned} \tag{A.46}$$

where g is the determinant of the metric tensor g_{ij}.

The divergence of a covariant vector A_i is defined as a divergence of its conjugate contravariant tensor (A.38):

$$A^i_{,i} = g^{ij} A_{j,i} \tag{A.47}$$

Definition A.3.7 Laplacian

A Laplace operator or a Laplacian of a scalar A is defined as

$$\Delta A \equiv g^{ik} A_{,ki} \tag{A.48}$$